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On the nonlinearity interpretation of q- and f-deformation and some applications

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Abstract. q-oscillators are associated to the simplest non-commutative example of Hopf algebra and may be considered to be the basic building blocks for the symmetry algebras of completely integrable theories. They may also be interpreted as a special type of spectral nonlinearity, which may be generalized to a wider class of f-oscillator algebras.

In the framework of this nonlinear interpretation, we discuss the structure of the stochastic process associated to q-deformation, the role of the q-oscillator as a spectrum-generating algebra for fast growing point spectrum, the deformation of fermion operators in solid-state models and the charge-dependent mass of excitations in f-deformed relativistic quantum fields.

1. Complete integrability, q-commutators and nonlinearity

In the last few years many papers have appeared concerning a deformation of the harmonic-oscillator algebra of creation and annihilation operators, called the q-oscillator algebra (Biedenharn 1989, Macfarlane 1989).

From a mathematical point of view, q-oscillators are associated to the simplest non-trivial example of Hopf algebra. However, the physical relevance of q-deformed creation and annihilation operators is not always very transparent in the studies that have been published on the subject. Therefore it is important to emphasize that there are, at least, two properties which make q-oscillators interesting objects for physics. The first is the fact that they naturally appear as the basic building blocks of completely integrable theories. Hence, in so far as complete integrability is important for physics, q-oscillators are a relevant physical tool. The second concerns the recently discovered connection between q-deformation and nonlinearity. In this paper we shall mainly be concerned with this second aspect. Nevertheless it is useful to emphasize the natural connection of the q-oscillator to complete integrability (Faddeev 1980, Izergin and Korepin 1982, Kulish and Reshetikhin 1983).

Associated with each solution of the Yang–Baxter equation there is a matrix algebra generated by the matrix elements of the Lax operator. The simplest non-trivial example of the R matrix leads to the matrix algebra of $SU_q(2)$

$$\begin{aligned}
&[S_3, S_\pm] = \pm S_\pm \\
&[S_+, S_-] = \frac{q^{2S_3} - q^{-2S_3}}{q - q^{-1}} = [2S_3]_q.
\end{aligned} \tag{1}$$

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By a generalization of the Jordan–Schwinger map this algebra may be realized in terms of creation and annihilation operators, namely

$$S_+ = A_1^\dagger A_2 \qquad S_- = A_2^\dagger A_1 \qquad S_3 = \tfrac{1}{2}(N_1 - N_2) \tag{2}$$

where $(A_1, A_1^\dagger)$ and $(A_2, A_2^\dagger)$ are mutually commuting boson operators satisfying

$$A_i A_i^\dagger - q^{-1} A_i^\dagger A_i = q^{N_i} \tag{3}$$

supplemented by the relations $[N_i, A_j^\dagger] = A_i^\dagger \delta_{ij}$, $[N_i, A_j] = -A_i\delta_{ij}$ and $A_i^\dagger A_i = [N_i]_q$. The last relation is equivalent to the requirement of invariance of the algebra under $q \longleftrightarrow q^{-1}$. This is called the algebra of the q-oscillator or the q-deformed Heisenberg algebra. The construction of q-deformed algebras, purely in terms of q-oscillators, may be extended to general $U_q(n)$. In this way, rather than a mathematical curiosity, q-oscillators appear as the very basic building blocks of completely integrable dynamical systems.

Another important property of q-oscillators is their relation to nonlinearity of a special type. In the remainder of this paper we shall deal with this aspect of q-deformation, in particular with the structure of the stochastic processes associated with q-oscillators, their hypothetical role as a spectrum generating algebra for fundamental excitations, the deformation of fermion operators in the construction of solid-state models and a relativistic generalization.

Using a nonlinear map (Polychronakos 1990, Curtright and Zachos 1990) the q-oscillator has been interpreted (Man'ko *et al* 1993a, b) as a nonlinear oscillator with a special type of nonlinearity which classically corresponds to an amplitude dependence of the oscillator frequency. This is seen as follows. Let

$$\begin{aligned} A &= af(N) \\ A^\dagger &= f(N)a^\dagger \end{aligned} \tag{4}$$

with

$$f(N) = \left(\frac{\sinh(\lambda N)}{N \sinh\lambda}\right)^{\frac{1}{2}} \tag{5}$$

where $q = \mathrm{e}^\lambda$. Then if $a, a^\dagger$ satisfy the usual undeformed commutation relations

$$[a, a^\dagger] = 1 \tag{6}$$

the operators $A, A^\dagger$ in (4) satisfy the q-deformed commutation relations (3). This means that the Hamiltonian

$$H = A^\dagger A = f(N)a^\dagger a f(N) \tag{7}$$

has a spectrum with the same structure as the spectrum of $a^\dagger a$, the difference being that the eigenvalues have values $nf^2(n)$, $n = 0, 1, 2, \ldots$, instead of n. Writing

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}(q + \mathrm{i}p) \\ a^\dagger &= \frac{1}{\sqrt{2}}(q - \mathrm{i}p) \end{aligned} \tag{8}$$

this means that the classical Hamiltonian

$$H = \tfrac{1}{2} f^2\{\tfrac{1}{2}(p^2 + q^2 - 1)\}(p^2 + q^2 - 1) \tag{9}$$

has as a solution, the oscillation

$$q = q_0 \cos \Omega t + \frac{1}{\Omega} p_0 \sin \Omega t$$
$$p = \frac{1}{\Omega} p_0 \cos \Omega t - q_0 \sin \Omega t \tag{10}$$

with

$$\Omega = f^2 \left(\frac{1}{2} q_0^2 + \frac{p_0^2}{2\Omega^2} - 1 \right) + \frac{1}{2} \left(q_0^2 + \frac{p_0^2}{\Omega^2} - 1 \right) f^{2'} \left(\frac{1}{2} q_0^2 + \frac{p_0^2}{2\Omega^2} - 1 \right). \tag{11}$$

Therefore the frequency is a function of the amplitude of the oscillation

$$\Omega = \Omega(p_0, q_0). \tag{12}$$

This being typical of nonlinear phenomena, it means that q-deformation is the quantum analogue of this type of nonlinearity.

As pointed out by Man'ko *et al* (1996), the association of nonlinearity to the deformation of the commutation relations may be generalized to relations of the form

$$AA^\dagger - g(N) A^\dagger A = h(N). \tag{13}$$

A solution of the type of equation (4) exists if

$$f^2(N+1)(1+N) - g(N) f^2(N) N = h(N). \tag{14}$$

f-deformed states, in the sense discussed above, appear as stationary states of a trapped laser-driven ion (Matos Filho and Vogel 1996).

Equation (14) establishes a general relation between deformation of the commutation relations and nonlinear modifications of the spectrum in the sense defined above. In the past, algebra deformation studies have been mostly concerned with the specific case of q-deformation because of its association to completely integrable systems. Nevertheless many results, including the construction of Hopf algebras (Polychronakos 1990), can be extended to the more general case of f-deformation defined in equation (14).

The type of quantum nonlinearity introduced by f-deformation provides a compact description of effects that are otherwise difficult to model. For example, the spectrum associated to q-deformation grows like $\sinh(\lambda n)$, that is, the local spacing grows with n, exponentially for large n. The relation of level spacings to the nature of the potential has been discussed by Vilela Mendes (1985), the conclusion being that a fast increasing level spacing cannot be obtained with reasonable local potentials. The discussion carried out was in connection with an outstanding problem of particle physics, namely the fact that the mass spectrum of the lepton and quark families grows quite remarkably. It is therefore interesting to find such a level spacing growth at the basic level of q-deformed creation and annihilation operators, the building blocks of completely integrable theories. For example, we might imagine the massive elementary leptons to be described by a composite operator product $C^\dagger A^\dagger$ of a fermion $C^\dagger$ and a q-boson $A^\dagger$ with excitations controlled by the Hamiltonian $H = kA^\dagger A + m_e$. Then the mass spectrum would be

$$m_n = k \frac{\sinh(\lambda n)}{\sinh \lambda} + m_e.$$

Identifying the muon and the tau with the $n = 1$ and 2 states leads to $k = 105$ MeV and $\lambda = 2.82$, which would imply for the next lepton excitation, if it exists, a mass of around 30 GeV.

2. Deformed fermion operators and solid-state models

In section 1 we have seen that, through a generalization of the Jordan–Schwinger map, the q-algebra $SU_q(2)$ may be realized with products of two mutually commuting q-boson operators (equation (2)). As in the undeformed case, a similar Jordan–Schwinger map exists for a pair $(C_1, C_1^\dagger)$ and $(C_2, C_2^\dagger)$ of mutually anticommuting q-deformed fermion operators.

$$S_+ = C_1^\dagger C_2 \qquad S_- = C_2^\dagger C_1 \qquad S_3 = \tfrac{1}{2}(N_1 - N_2) \tag{15}$$

with

$$C_i C_i^\dagger + q C_i^\dagger C_i = q^{N_i} \tag{16}$$

and $[N_i, C_j^\dagger] = C_i^\dagger \delta_{ij}$, $[N_i, C_j] = -C_i \delta_{ij}$, $C_i^\dagger C_i = [N_i]_q$. Hence q-fermions may also be considered as building blocks for quantum algebras. As in the boson case there is a nonlinear interpretation for fermion q- and f-deformation. Let

$$\begin{aligned} C &= c\overline{f}(N) \\ C^\dagger &= \overline{f}(N)c^\dagger. \end{aligned} \tag{17}$$

Then the C operators will satisfy

$$CC^\dagger + \overline{g}(N)C^\dagger C = \overline{h}(N) \tag{18}$$

if

$$\overline{f}^2(N+1)(1-N) + \overline{g}(N)\overline{f}^2(N)N = \overline{h}(N). \tag{19}$$

For the case of q-deformation (equation (16)), $\overline{g}(N) = q$, $\overline{h}(N) = q^N$, and the solution of equation (19) with the limit $\overline{f} \to 1$ when $q \to 1$ is

$$\overline{f}(N) = q^{(N-1)/2}. \tag{20}$$

The nonlinear representation (17) means that the deformation is equivalent to the introduction of an occupation number dependence on the action of the operators. This is a convenient tool to generate effects of this type in physical models. As an example consider the Hubbard model (Hubbard 1963), a paradigmatic model for the problem of electron correlations. The Hamiltonian is

$$H = -\sum_{\sigma,\langle x,y\rangle} t c_{x\sigma}^\dagger c_{y\sigma} + \sum_x U_x (N_{x\uparrow} - \tfrac{1}{2})(N_{x\downarrow} - \tfrac{1}{2}) \tag{21}$$

the first sum being over nearest-neighbour lattice sites $\langle x, y\rangle$ and $\sigma \in \{\uparrow, \downarrow\}$ the electron polarization. f-deformation of this model means that the electron operators $c_{x\sigma}^\dagger$, $c_{y\sigma}$ are to be replaced by deformed operators

$$\begin{aligned} C_{x\sigma}^\dagger &= \overline{f}(N_x)c_{x\sigma}^\dagger \\ C_{x\sigma} &= c_{x\sigma}\overline{f}(N_x) \end{aligned} \tag{22}$$

with $N_x = N_{x\uparrow} + N_{x\downarrow}$. The second (Coulomb) term in (21) is unchanged and the hopping term becomes

$$-\sum_{\sigma,\langle x,y\rangle} t\overline{f}(N_x)\overline{f}(N_y+1)c_{x\sigma}^\dagger c_{y\sigma}. \tag{23}$$

It means that the hopping amplitude now depends on whether there are other electrons (besides the one that is hopping) in the two sites involved in the hopping. There is reasonable evidence that such an effect is indeed present in oxide superconductors and what we have done is to interpret the Hirsch model (Hirsch 1992, 1994) as an f-deformation of the Hubbard model.

3. Deformed white noise and Brownian motion

There is a canonical association of the time dependence of harmonic-oscillator spectral modes with Brownian motion. This is most clearly seen in the Paley–Wiener construction of Brownian motion (Paley and Wiener 1934, Hida 1980). Using the Paley–Wiener construction with the spectrum of $H = A^\dagger A$, instead of $a^\dagger a$, one obtains the corresponding q- or f-deformed process.

Let $X_k(\omega)$ and $Y_k(\omega)$, $k = 0, \pm 1, \pm 2, \ldots,$ be a sequence of independent identically distributed normalized Gaussian random variables. Then

$$Z_k(\omega) = \tfrac{1}{2}(X_k(\omega) + \mathrm{i}Y_k(\omega)) \tag{24}$$

is a sequence of complex Gaussian random variables. The formal sum

$$\eta(t,\omega) = \sum_{n=-\infty}^{\infty} Z_k(\omega)\mathrm{e}^{\mathrm{i}f^2(n)nt} \tag{25}$$

is not convergent as a L^2 random variable, but it has a meaning in a generalized function sense to be made precise below. By construction

$$\langle \eta(t,\omega)\rangle = 0 \tag{26}$$

and using (5) the covariance is

$$\langle \eta(t,\omega)\eta(0,\omega)\rangle_q = \sum_{n=0}^{\infty} \frac{t^n}{n!}\left(\frac{\sin(\lambda\partial_t)}{\sinh\lambda} - \partial_t\right)^n \delta(t) \tag{27}$$

where we have used the $f^2(n)$ function appropriate for q-deformation. Using $t^n\delta^{(k)}(t) = (-1)^n n!\binom{k}{n}\delta^{(k-n)}(t)$, equation (27) may be converted into a multipole series. Therefore $\eta(t,\omega)$ is a generalized random process in an ultradistribution sense (Sebastião e Silva 1967, Hoskins and Sousa Pinto 1994). For small λ, $(q \simeq 1)$, it becomes

$$\langle \eta(t,\omega)\eta(0,\omega)\rangle_q \simeq \left(1 + \frac{\lambda^2}{3!}\right)\delta(t) + \frac{\lambda^2}{2}\delta''(t). \tag{28}$$

The general expression for the characteristic functional of q-deformed 'white' noise is

$$C(\xi) = \exp\left\{-\frac{1}{2}\sum_{n=0}^{\infty}\int \mathrm{d}t\,\mathrm{d}s\,\xi^*(t)\frac{(t-s)^n}{n!}\left(\frac{\sin(\lambda\partial_t)}{\sinh\lambda} - \partial_t\right)^n \delta(t-s)\xi(s)\right\} \tag{29}$$

with the corresponding deformed Brownian motion obtained by integration

$$X(t,\omega) = \int_0^t \eta(s,\omega)\,\mathrm{d}s. \tag{30}$$

Note that in the Paley–Wiener construction of the deformed stochastic processes, the unequal time correlations are generated by the choice of the spectrum $nf^2(n)$, that is, by the choice of the Hamiltonian $A^\dagger A$ as defining the time dependence of the oscillation modes. This is the construction that captures the interpretation of q-deformation as a form of spectral nonlinearity. In Man'ko and Vilela Mendes (1993) a different construction was discussed where, by using the isomorphism between Brownian motion and boson fields, we have used the commutation relations (3) but imposed delta correlation for the process. In that case the covariance is Gaussian, although higher-order correlations are not, differing from the

Gaussian ones by a kind of braiding structure arising from the commutation relations. The construction of Man'ko and Vilela Mendes (1993) as well as a similar one developed by Bozejko and Speicher (1991) starting from a different set of (quon) commutation relations, are also perfectly consistent. However, the one presented in this paper, because of its direct interpretation in terms of nonlinearity of the dynamics, is probably more interesting for its physical consequences.

4. A relativistic generalization. Quantum fields

Another use of the q-nonlinear (f-nonlinear) interpretation of deformations is the possibility to write down equations for quantum fields incorporating nonlinearity. An attempt to describe classical deformed scalar fields, using the number of quanta as an integral of motion, has been discussed by Man'ko *et al* (1995). However, for relativistic quantum fields, the number of quanta is not preserved and we shall use another constant of motion.

Let $a_+^\dagger(k)$ $a_-^\dagger(k)$ $a_+(k)$ $a_-(k)$ be creation and annihilation operators for charged free bosons of three-momentum k and rest mass m_0. The relativistic Hamiltonian of the free boson field is

$$H_0 = \sum_k k^0\{a_+^\dagger(k)a_+(k) + a_-^\dagger(k)a_-(k)\} \tag{31}$$

with

$$k^0 = \sqrt{k^2 + m_0^2}. \tag{32}$$

Now define the operators

$$\begin{aligned} A_\pm(k) &= a_\pm(k) f(k, Q) \\ A_\pm^\dagger(k) &= f(k, Q) a_\pm^\dagger(k) \end{aligned} \tag{33}$$

with

$$f(k, Q) = \left(\frac{k^2 + M^2(Q)}{k^2 + m_0^2}\right)^{\frac{1}{2}} \tag{34}$$

$M^2(Q)$ being an arbitrary function of the charge operator Q

$$Q = \int \mathrm{d}^3k\, \{a_+^\dagger(k)a_+(k) - a_-^\dagger(k)a_-(k)\} \tag{35}$$

which for charged free boson fields is a constant of motion related to the Noether current

$$j^\mu = \mathrm{i}(\phi^*\nabla^\mu\phi - \phi\nabla^\mu\phi^*) \tag{36}$$

by

$$Q = \mathrm{i}\int \mathrm{d}^3x\, (\phi^*\, \dot{\phi} - \phi\, \dot{\phi}^{\,*}). \tag{37}$$

If in the Hamiltonian (31) we replace $a_\pm(k)$ by $A_\pm(k)$ we obtain

$$H = \sum_k k^0\{A_+^\dagger(k)A_+(k) + A_-^\dagger(k)A_-(k)\} \tag{38}$$

which, by construction, represents a relativistic quantum system where the mass of the excitations created by $A^\dagger_\pm(k)$ depends on the pre-existing total charge. On the other hand this change in the dynamical structure of the excitations may be interpreted, as before, as a change on the commutation relations of the creation and annihilation operators, namely

$$\begin{aligned}
&A_\pm(k)A^\dagger_\pm(k') - \frac{f(k,Q\pm 1)f(k',Q\pm 1)}{f(k,Q)f(k',Q)}A^\dagger_\pm(k')A_\pm(k) = f^2(k,Q\pm 1)\delta(k-k') \\
&A_+(k)A^\dagger_-(k') - \frac{f(k,Q+1)f(k',Q+1)}{f(k,Q+2)f(k',Q)}A^\dagger_-(k')A_+(k) = 0 \\
&A_-(k)A^\dagger_+(k') - \frac{f(k,Q-1)f(k',Q-1)}{f(k,Q-2)f(k',Q)}A^\dagger_+(k')A_-(k) = 0.
\end{aligned} \tag{39}$$

5. Conclusions

Deformation of the canonical commutation relations in the sense of equation (13) may, through equation (4), be interpreted as a modification of the associated spectrum which is the quantum analogue of the classical nonlinear modification of the frequency of oscillation. q-deformations are a particular type of such deformations which have an important role because of its association with completely integrable theories.

As illustrated in sections 1, 2 and 4, the spectral interpretation of the algebraic deformation of boson and fermion operators provides simple models for physical systems with non-uniform or density-dependent spectra.

In addition, the nonlinear modification of the spectrum arising from deformation of the algebra has interesting implications in other structures, for example in the construction of the associated stochastic processes. By an extension of the tools developed in white noise analysis (Hida *et al* 1993) these deformed processes may provide a natural infinite-dimensional analysis framework for nonlinear interacting systems.

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